§ 1 Introduction to the Concept of Analytic Function

With the stereographic projection, we have just seen, without clearly stating it, our first example of a function which takes real numbers as inputs, and outputs a complex number.

As we move on to studying functions and their properties, 4 cases may in principle be considered: real functions of real variables, real functions of complex variables, complex functions of real variables, and complex functions of complex variables. Fortunately, the vast majority of concepts we will apply to functions can be defined in the same way in the 4 cases. This is because the concept of limit can be defined in the same way in all 4 cases.

1.1 Limits and Continuity

Definition 1. A function f has the *limit* L (L finite) as z tends to z_0, written $\lim_{z\to z_0} f(z) = L$, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(z) - L| < \varepsilon$ for all z such that $|z - z_0| < \delta$.

You can observe that this indeed agrees with the definition you have all seen for real functions of real variables. If the input z is complex, or the output $f(z)$ is complex, what used to be the absolute value should now be understood as the modulus.

Similar definitions are easily constructed for the cases in which L is infinite, as you have done for real variables.

- As we have seen in the previous lecture, for two complex numbers z_1 and z_2, $|z_1 z_2| = |z_1|\,|z_2|$ and $|z_1 + z_2| \le |z_1| + |z_2|$ so recalling the proofs in the real variables case, we easily see that the limit laws (sum law, product law) also hold in the complex case.
- Since $|z| = |\bar{z}|$ for any $z \in \mathbb{C}$, if $\lim_{z\to z_0} f(z) = L$, then $\lim_{z\to z_0} \overline{f(z)} = \bar{L}$
- Combining the previous two results,

$$\begin{cases} \lim_{z\to z_0} Re\,(f(z)) & = Re\,(L) \\ \lim_{z\to z_0} Im\,(f(z)) & = Im\,(L) \end{cases} \tag{1}$$

which can be seen as an alternative way of defining the limit of $f(z)$.

Definition A function f is *continuous* at z_0 if $\lim_{z\to z_0} f(z) = f(z_0)$.

This is again the same definition as in the case of real variables so we know that we could easily prove that the sum and product of continuous functions are continuous functions.

From the definition of the limit, we can also conclude that if f is continuous at z_0, then so is $\bar{f}$, $Re\,(f)$ and $Im\,(f)$.

Definition A function f is *(complex) differentiable* at z_0 if

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. This number is called the *derivative* of f at z_0, written

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Once more, the definition is standard. However, depending on the case considered - real function or complex function, real variable or complex variable - the existence of a derivative can have far-reaching consequences regarding the properties of the function.

• Let us start with the simplest case: a complex function f of a real variable x. One may write

$$f(x) = u(x) = iv(x)$$

f has a derivative $f'(x_0)$ at x_0 if and only if u and v are differentiable at x_0, and

$$f'(x_0) = u'(x_0) + iv'(x_0)$$

• Consider now a *real-valued* function f of a *complex* variable z. If f is differentiable at z_0, then

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$$

This is in particular true along the horizontal line $z = z_0 + h$, $h \in \mathbb{R}$:

$$\lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h} = f'(z_0)$$

$f'(z_0)$ is therefore a real number.

But z can also approach z_0 along the vertical line $z = z_0 + ih$, $h \in \mathbb{R}$:

$$\lim_{h\to 0} \frac{f(z_0 + ih) - f(z_0)}{ih} = f'(z_0)$$

from which we conclude that $f'(z_0)$ is also purely imaginary, and thus zero.

We proved the following result: *If a real-valued function of a complex variable is differentiable at a point, then its derivative is zero at this point.*

• For complex functions of complex variables, differentiability has fundamental consequences for the properties of the function. We now devote the next section (and many more in the remainder of this course) to this crucial case.

Cauchy-Riemann Equations

For any $z \in \mathbb{C}$, let us write $z = x + iy$, with $(x, y) \in \mathbb{R}^2$, and $f(z) = u(x, y) + iv(x, y)$. where u and v are real valued functions.

If f is complex differentiable at z_0, then $f'(z_0) = \lim\limits_{z\to 0} \left(f(z + z_0) - f(z_0)\right)/z$ exists, independently of the path that z follows towards 0 in the complex plane. In particular,

$$\begin{aligned}
f'(z_0) &= \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{f(z_0 + h) - f(z_0)}{h} \\
&= \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{u(x_0 + h, y_0) + iv(x_0 + h, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{h} \\
&= \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \left[\frac{u(x_0 + h, y_0) - u(x_0, y_0)}{h} + i\,\frac{v(x_0 + h, y_0) - v(x_0, y_0)}{h}\right] \\
&= \frac{\partial u}{\partial x}(x_0, y_0) + i\,\frac{\partial v}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial x}(z_0) \qquad (2)
\end{aligned}$$

But we can also write

$$\begin{aligned}
f'(z_0) &= \lim_{\substack{z\to 0\\ z\in i\mathbb{R}}} \frac{f(z_0 + z) - f(z_0)}{z} \\
&= \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{u(x_0, y_0 + h) + iv(x_0, y_0 + h) - u(x_0, y_0) - iv(x_0, y_0)}{ih} \\
&= \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \left[\frac{u(x_0, y_0 + h) - u(x_0, y_0)}{ih} + i\,\frac{v(x_0, y_0 + h) - v(x_0, y_0)}{ih}\right]
\end{aligned}$$

$$= \frac{1}{i}\left[\frac{\partial u}{\partial y}(x_0,\, y_0) + i\frac{\partial v}{\partial y}(x_0,\, y_0)\right] = -i\frac{\partial u}{\partial y}(x_0,\, y_0) + \frac{\partial v}{\partial y}(x_0,\, y_0) = \frac{1}{i}\frac{\partial f}{\partial y}(z_0) \quad (3)$$

Equating (2) and (3), we must have

$$\begin{cases} \dfrac{\partial u}{\partial x} &= \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} &= -\dfrac{\partial v}{\partial x} \end{cases} \iff \frac{\partial f}{\partial y} = i\frac{\partial f}{\partial x} \iff \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) = 0 \iff \frac{\partial f}{\partial \bar{z}} = 0 \quad (4)$$

These are the well-known *Cauchy-Riemann equations*. **If a function f is complex differentiable, then its real and imaginary parts satisfy the Cauchy-Riemann equations.**

Conversely, let us assume $f(z) = u(x,\, y) + iv(x,\, y)$ with u and v real valued functions which have continuous first-order partial derivatives which satisfy the Cauchy-Riemann equations. Then, one may expand

$$\begin{aligned} u(x_0 + h,\, y_0 + k) &= u(x_0,\, y_0) + \frac{\partial u}{\partial x}(x_0,\, y_0)h + \frac{\partial u}{\partial y}(x_0,\, y_0)k + \varepsilon_1 h + \varepsilon_2 k \\ v(x_0 + h,\, y_0 + k) &= v(x_0,\, y_0) + \frac{\partial v}{\partial x}(x_0,\, y_0)h + \frac{\partial v}{\partial y}(x_0,\, y_0)k + \varepsilon_3 h + \varepsilon_4 k \end{aligned} \quad (5)$$

where $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$, $\varepsilon_3 \to 0$, $\varepsilon_4 \to 0$ as $(h,\, k) \to (0,\, 0)$. Hence,

$$\lim_{h+ik\to 0} \frac{f(z_0 + h + ik) - f(z_0)}{h + ik} = \lim_{(h,\,k)\to(0,\,0)} \frac{\dfrac{\partial u}{\partial x}h + \dfrac{\partial u}{\partial y}k + i\left(\dfrac{\partial v}{\partial x}h + \dfrac{\partial v}{\partial y}k\right)}{h + ik}$$

Using the Cauchy-Riemann equations, this becomes

$$\lim_{(h,\,k)\to(0,\,0)} \frac{\left(\dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}\right)(h + ik)}{h + ik} = \frac{\partial u}{\partial x}(x_0,\, y_0) + i\frac{\partial v}{\partial x}(x_0,\, y_0) = \frac{\partial f}{\partial x}(z_0)$$

We conclude that f is complex differentiable. □

We have just proved the following theorem:

Theorem $f(z) = u(x,\, y) + iv(x,\, y)$ is complex differentiable with continuous derivative $f'(a)$ at a iff $u(x,\, y)$ and $v(x,\, y)$ have continuous first-order partial derivatives which satisfy the Cauchy-Riemann equations.

The results above give us explicit ways to write $f'(z)$ in terms of the real and imaginary parts of f :

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

Example Let f be defined by

$$f(x, y) = u(x, y) + iv(x, y) = \begin{cases} \dfrac{xy(x + iy)}{x^2 + y^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0. \end{cases}$$

Since $f(x,\, 0) = f(0,\, y) = 0$ for all $x,\, y \in \mathbb{R}$, we have $f_x(0,\, 0) = f_y(0,\, 0) = 0$ and $u_x(0,\, 0) = u_y(0,\, 0) = v_x(0,\, 0) = v_y(0,\, 0) = 0$, so the Cauchy Riemann equations hold for u and v at $(0,\, 0)$.

However, note that u_x, u_y, v_x, v_y are not continuous at $(0, 0)$, and if we let $z = x + iy \to 0$ along the line $y = \alpha x$, $\alpha \in \mathbb{R}$, then

$$\lim_{\substack{z = x+iy \to 0 \\ y = \alpha x}} \frac{f(z) - f(0)}{z} \overset{y=\alpha x}{=} \lim_{(x,y)\to(0,0)} \frac{xy(x+iy)}{x^2+y^2} \overset{y=\alpha x}{=} \frac{\alpha}{1+\alpha^2} \quad \text{depends on } \alpha,$$

which implies that the limit does not exist and f is not differentiable at $z = 0$.

1.2 Analytic Functions

Definition A complex function f of a complex variable z is *analytic at z_0* (or *holomorphic at z_0*) if f is *differentiable in a neighborhood of z_0*. Similarly, f is analytic on a set S if f is differentiable at all points of some open set containing S.

Remarks

(1) If a function f is analytic at z_0, it is continuous at z_0. Indeed,

$$\lim_{z\to 0} [f(z+z_0) - f(z_0)] = \lim_{z\to 0} [z f'(z_0)] = 0$$

(2) If f and g are two functions that are analytic at z_0, then

- so is their sum $f + g$ with $(f+g)'(z_0) = f'(z_0) + g'(z_0)$,
- so is their product fg with $(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$,
- so is their quotient f/g with

$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2} \quad \text{provided } g(z_0) \neq 0.$$

(3) If f is analytic at z_0, and g is analytic at $w_0 = f(z_0)$, then $g \circ f$ is analytic at z_0, and $(g \circ f)'(z_0) = f'(z_0) g'(f(z_0))$

Example It is easy to verify that $f(z) = z$ is analytic on $\mathbb{C}$, and that $g(z) = 1 + z^2$ is analytic on $\mathbb{C}$ and nonzero on $\mathbb{C} \setminus \{-i, i\}$. Hence $h(z) = z/(1+z^2)$ is analytic on $\mathbb{C} \setminus \{-i, i\}$.

Example $f(z) = z^2 = x^2 - y^2 + 2ixy$ is analytic on all of $\mathbb{C}$ since

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}$$

and u_x, u_y, v_x, v_y are clearly continuous on $\mathbb{R}^2$.

$$f'(z) = \frac{\partial f}{\partial z} = 2z = 2x + 2iy = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$$

Harmonic Functions

Let $f(z) = u(x, y) + iv(x, y)$ be an analytic function. Then u and v satisfy the Cauchy-Riemann equations:

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$$

Let us take for granted, for the time being, that u and v have continuous higher order partial derivatives (we will prove this later in this course). We can write

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial x \partial y} = 0$$
$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial x \partial y} = 0$$

In other words, $\Delta u = 0$, $\Delta v = 0$. The real and imaginary parts of an analytic function satisfy Laplace's equation. They are *harmonic functions*.

If two harmonic functions u and v satisfy the Cauchy-Riemann equations, then we say that v is a *conjugate harmonic* function of u.

Example It is easy to see that $f(z) = z^3$ is analytic on $\mathbb{C}$. We can write $u(x, y) = x^3 - 3y^2 x$, $v(x, y) = 3x^2 y - y^3$, and compute

$$\Delta u = 6x - 6x = 0 \quad , \quad \Delta v = 6y - 6y = 0$$

1.3 Polynomials

$f : z \mapsto z$ is analytic on all of $\mathbb{C}$ since its real and imaginary parts satisfy the Cauchy-Riemann relations and have continuous first-order partial derivatives for all $(x, y) \in \mathbb{R}^2$.

A function f which is analytic on all of $\mathbb{C}$ is called an *entire* function. For example, $f(z) = z$ is an entire function.

Considering our result for the sum and product of analytic functions, this means that for $(a_0, a_1, \ldots, a_N) \in \mathbb{C}^{N+1}$, the polynomial

$$P(z) = \sum_{i=0}^{N} a_i z^i$$

is also an entire function. From our result for the derivative of the product of functions, for any $a \in \mathbb{C}$, we may write

$$P'(a) = \sum_{i=1}^{N} i a_i z^{i-1}$$

• In this course, we will soon prove the *fundamental theorem of algebra: Every polynomial P of a complex variable has a root.*

Using this result without proof for the time being, we can say that any polynomial of degree N can be written as

$$P(z) = a_N (z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_N)$$

where the α_i may or may not be distinct.

The α_i's are called the *zeros* of P. If k of the α_i coincide, we say that α_i is a *zero of order k.* A zero of order 1 is called a *simple zero*. If α_i is a *zero of order k*, one may write $P(z) = (z - \alpha_i)^k P_k(z)$, with P_k a polynomial such that $P_k(\alpha_i) \neq 0$. Thus, $P(\alpha_i) = P'(\alpha_i) = \cdots = P^{(k-1)}(\alpha_i) = 0$, but $P^{(k)}(\alpha_i) \neq 0$: the order of a zero equals the order of the first nonvanishing derivative.

Theorem 1. (Gauss-Lucas Theorem) The smallest convex polygon that contains the zeros of $P(z)$ also contains the zeros of $P'(z)$.

Proof Let us consider a zero a of $P'(z)$. If a is also a zero of $P(z)$, there is nothing to prove. If a is not a zero of $P(z)$, we can write

$$\frac{P'(a)}{P(a)} = \sum_{i=1}^{N} \frac{1}{a-\alpha_i} = 0 \iff \sum_{i=1}^{N} \frac{\bar{a}-\overline{\alpha_i}}{|a-\alpha_i|^2} = 0 \iff \sum_{i=1}^{N} \frac{a-\alpha_i}{|a-\alpha_i|^2} = 0$$

$$\iff \sum_{i=1}^{N} \frac{a}{|a-\alpha_i|^2} = \sum_{i=1}^{N} \frac{\alpha_i}{|a-\alpha_i|^2} \iff a = \left(\sum_{i=1}^{N} \frac{1}{|a-\alpha_i|^2}\right)^{-1} \sum_{i=1}^{N} \frac{\alpha_i}{|a-\alpha_i|^2}$$

a can thus be viewed as the barycenter of the roots of α_i of $P(z)$ with positive coefficients that sum to one (i.e. a convex combination of all the roots of $P(z)$).This completes our proof. □

Corollary If all zeros of a polynomial $P(z)$ lie in a half plane, then all zeros of the derivative $P'(z)$ lie in the same half plane.

1.4 Rational Functions

Consider two polynomials $P(z)$ and $Q(z)$ which do not have common zeros. Then the rational function

$$R(z) = \frac{P(z)}{Q(z)}$$

is analytic away from the zeros of $Q(z)$.

The zeros of $Q(z)$ are called *poles* of $R(z)$, and the *order of a pole* is equal to the order of the corresponding zero of $Q(z)$.

Counting Poles and Zeros

Definition Rational functions are often given the value ∞ at the zeros of $Q(z)$. It must therefore be considered as a function with values in the extended complex plane $\widehat{\mathbb{C}}$, and as such it is continuous. To do this, one considers the function $R_\infty(z) = R\left(\frac{1}{z}\right)$.

If $R_\infty(0) = 0$, then $R(z)$ is said to have a *zero* at ∞.

If $R_\infty(0) = \infty$, then $R(z)$ is said to have a *pole* at ∞.

Example Let $R(z) = z$, $R_\infty(z) = \frac{1}{z}$, so $R(z)$ has a pole of order 1 at ∞.

Now, let us use $R_\infty(z)$ to count the number of poles and of zeros of an arbitrary rational function

$$R(z) = \frac{a_N z^N + a_{N-1}z^{N-1} + \cdots + a_1 z + a_0}{b_M z^M + b_{M-1}z^{M-1} + \cdots + b_1 z + b_0} \implies R_\infty(z) = z^{M-N} \frac{a_N + a_{N-1}z + \cdots + a_1 z^{N-1} + a_0 z^N}{b_M + b_{M-1}z + \cdots + b_1 z^{M-1} + b_0 z^M}$$

with N finite zeros and M, and

- if $N < M$, then $R(z)$ has a zero of order $M-N$ at ∞, and

$$\begin{cases} \text{the total number of zeros (counting the order of each zero) is } M-N+N = M = \max(N,\, M) \\ = \text{ the total number of poles is } M = \max(N,\, M) \end{cases}$$

- if $N > M$, then $R(z)$ has a pole of order $N-M$ at ∞, and

$$\begin{cases} \text{the total number of zeros is } N = \max(N,\, M) \\ = \text{ the total number of poles is } M+N-M = N = \max(N,\, M) \end{cases}$$

- if $N = M$, then

$$R_\infty(0) = \frac{a_N}{b_M} \begin{cases} \neq 0 \\ \neq \infty \end{cases}$$

i.e. $R(z)$ has neither a pole nor a zero at ∞, and

$$\begin{cases} \text{the total number of zeros is } N = M = \max(N,\, M) \\ = \ \text{the total number of poles is } M = N = \max(N,\, M) \end{cases}$$

Hence, we have proved the following theorem.

Theorem Let $P(z)$, $Q(z)$ be polynomials without common zeros, and let $R(z) = P(z)/Q(z)$ be a rational function. The total number of zeros and the total number of poles of $R(z)$ are equal. This common number of zeros and poles ($= \max\{\deg P,\, \deg Q\}$) is called the *order* of $R(z)$.

Remark If $R(z)$ is a rational function of order p, then for any $a \in \mathbb{C}$, the equation $R(z) = a$ has exactly p roots, since $\tilde{R}(z) = R(z) - a$ is a rational function with the same poles as $R(z)$.

Example A rational function of order 1 is a *linear fraction*

$$R(z) = \frac{az + b}{cz + d} \quad \text{with } ad - bc \neq 0.$$

Such rational function is often called a *linear fractional transformation*, or *Möbius transform*.

From the theorem, we know that for any $w \in \mathbb{C}$, $R(z) = w$ has a unique solution, which is

$$z = R^{-1}(w) = \frac{dw - b}{-cw + a}$$

Partial Fractions

• Let us consider a rational function $R(z) = \dfrac{P(z)}{Q(z)}$ such that $R(z)$ has a pole at ∞. By Euclidean division of $P(z)$ by $Q(z)$, one can write

$$R(z) = G(z) + H(z)$$

where $G(z)$ is a polynomial without constant term and $H(z)$ is a rational function for which the degree of the numerator is at most equal to that of the denominator, so that $H(\infty)$ is finite.

The degree of $G(z)$ is the order of the pole of $R(z)$ at ∞, and the polynomial $G(z)$ is called the *singular part of $R(z)$ at ∞*.

• Let the distinct *finite* poles of $R(z)$ be denoted by $\beta_1, \beta_2, \ldots, \beta_k$.

The function $R_j(\zeta) = R\left(\beta_j + \dfrac{1}{\zeta}\right)$ is a rational function ζ with a pole at $\zeta = \infty$. We can decompose it as

$$R_j(\zeta) = G_j(\zeta) + H_j(\zeta)$$

just as we did for $R(z)$ before, with $H_j(\zeta)$ finite at $\zeta = \infty$.

By the substitution $z = \beta_j + \dfrac{1}{\zeta} \implies \zeta = \dfrac{1}{z - \beta_j}$, we then have

$$R(z) = R_j\left(\frac{1}{z - \beta_j}\right) = G_j\left(\frac{1}{z - \beta_j}\right) + H_j\left(\frac{1}{z - \beta_j}\right)$$

where $G_j\left(\frac{1}{z-\beta_j}\right)$ is a polynomial in $\frac{1}{z-\beta_j}$ without constant term, and is the *singular part of $R(z)$ at β_j*, and $H_j\left(\frac{1}{z-\beta_j}\right)$ is finite for $z=\beta_j$.

- Consider now the function

$$F(z) := R(z) - G(z) - \sum_{j=1}^{k} G_j\left(\frac{1}{z-\beta_j}\right)$$

$F(z)$ can only have poles at $\beta_1, \beta_2, \ldots, \beta_k$ and ∞. Furthermore, by construction $F(z)$ is finite at $\beta_1, \beta_2, \ldots, \beta_k$ and ∞, since $H(z)$ is finite at $z=\infty$ and $H_j\left(\frac{1}{z-\beta_j}\right)$ is finite at β_j for each $1 \le j \le k$. Since $F(z)$ is a rational function which is finite everywhere, it must be a constant. Absorbing this constant inside $G(z)$, which we then call $\widetilde{G}(z)$, and obtain

$$R(z) = \widetilde{G}(z) + \sum_{j=1}^{k} G_j\left(\frac{1}{z-\beta_j}\right) \tag{6}$$

The construction above demonstrates rigorously that *every rational function has a representation by partial fractions*. You have certainly already used partial fractions as a technique to compute integrals of rational functions.

§ 2 Elementary Theory of Power Series

The most natural way to define the next class of analytic functions (exp, cosh, sinh, cos, sin,...) is through power series. We start by looking at some key properties of power series.

Radius of Convergence

Theorem 2. Consider the sequence $\{c_n\}_{n=0}^{\infty}$ of complex numbers. There exists $0 \le R \le \infty$ such that

$$\begin{cases} \sum_{n=0}^{\infty} c_n z^n \text{ converges absolutely for } 0 \le |z| < R, \\ \sum_{n=0}^{\infty} c_n z^n \text{ diverges for } |z| > R. \end{cases}$$

This number R, obviously unique, is called *the radius of convergence of the power series*, and is given by

$$R = \frac{1}{\limsup_{n\to\infty} |c_n|^{1/n}}, \quad \text{where} \quad \limsup_{n\to\infty} |c_n|^{1/n} := \lim_{n\to\infty}\left(\sup_{m\ge n} |c_m|^{1/m}\right) \tag{7}$$

Proof

- If $R=+\infty$, then $\limsup_{n\to\infty} |c_n|^{1/n} = 0$ and $\limsup_{n\to\infty} |c_n|^{1/n}|z| = 0$ for all $z \in \mathbb{C}$. Thus, for each $z \in \mathbb{C}$, $\exists N \in \mathbb{N}$ such that $\forall n > N$ implies $|c_n z^n| \le 1/2^n$, so that $\sum_{n=0}^{\infty} |c_n z^n|$ converges for all z.
- If $R = 0$, then $\limsup_{n\to\infty} |c_n|^{1/n} = \infty$. Thus, for any $z \ne 0$, there exists infinitely many values of n such that $|c_n|^{1/n} \ge 1/|z|$. Hence $|c_n z^n| \ge 1$, the terms of the series do not approach zero, and the series diverges. (The fact that the sereis converges for $z=0$ is obvious.)

- If $0 < R < +\infty$, then $\limsup_{n\to\infty} |c_n|^{1/n} = 1/R$. Thus, $\forall\,\epsilon > 0$, $\exists\, N \in \mathbb{N}$ such that

$$\text{if } n \geq N \implies |c_n| < \left(\frac{1}{R} + \varepsilon\right)^n$$

Let z be such that $0 \leq |z| < R$. Then $|z|/R < 1$, so $\exists\,\varepsilon$ such that

$$|z|\left(\frac{1}{R} + \varepsilon\right) < 1$$

For this ε, there exists $M \in \mathbb{N}$ such that

$$\text{if } n \geq M \implies \sum_{n=M}^{\infty} |c_n z^n| < \sum_{n=M}^{\infty} \left[\left(\frac{1}{R} + \varepsilon\right)|z|\right]^n$$

The series is dominated by a convergence geometric series, so it converges.
Conversely, if $|z| > R$, $\exists\,\varepsilon$ such that

$$|z|\left(\frac{1}{R} - \varepsilon\right) > 1$$

By definition of R, for this ε there exists a subsequence $\{c_{n_k}\}_{k=1}^{\infty}$ for which

$$|c_{n_k}| > \left|\frac{1}{R} - \varepsilon\right|^{n_k} \implies |c_{n_k} z^{n_k}| > \left(|z|\left|\frac{1}{R} - \varepsilon\right|\right)^{n_k} \quad \text{for each } k \in \mathbb{N}$$

Since the terms on the right-hand side of the inequality are unbounded, our series diverges. □

Power series and analyticity

Theorem The power series $f(z) = \sum_{n=0}^{\infty} c_n z^n$ is analytic on the open disk $D_R(0)$ (i.e. the set $0 \leq |z| < R$) of convergence, and can be differentiated termwise in the disc of convergence:

$$f'(z) = \sum_{n=1}^{\infty} n c_n z^{n-1} \tag{8}$$

That series has the same radius of convergence as f.

Proof

- Since $|nc_n|^{1/n} = |c_n|^{1/n} e^{\ln n/n}$ for each $n \in \mathbb{N}$, and $\lim_{n\to\infty} e^{\ln n/n} = 1$, the two series indeed have the same radius of convergence.
- Let $z_0 \in D_R(0)$ and $r \in \mathbb{R}$ such that $|z_0| < r < R$. $\forall\, n \in \mathbb{N}$, the function

$$u_n(z) = c_n\left(z^{n-1} + z_0 z^{n-2} + \cdots + z_0^{n-1}\right)$$

is continuous on $D_R(0)$ and on this disk satisfies

$$\sup_{z\,\in\, D_r(0)} |u_n(z)| \leq |c_n| n r^{n-1}$$

with $\sum_{n=1}^{\infty} n|c_n|r^{n-1}$ finite, as we proved in the point above.

We conclude that $\sum_{n=1}^{\infty} u_n(z) = \lim_{N\to\infty} \sum_{n=1}^{N} u_n(z)$ converges uniformly on $\overline{D_r(0)}$ for all $0 < r < R$, and its sum is continuous on this disk, in particular at z_0.
Finally, we observe that $\forall\, z \in D_r(0) \setminus \{z_0\}$,

$$\sum_{n=1}^{\infty} u_n(z) = \sum_{n=1}^{\infty} c_n \left(z^{n-1} + z_0 z^{n-2} + \cdots + z_0^{n-1}\right) = \sum_{n=1}^{\infty} c_n \frac{z^n - z_0^n}{z - z_0} = \frac{f(z) - f(z_0)}{z - z_0}$$

and that

$$\sum_{n=1}^{\infty} n c_n z_0^{n-1} = \sum_{n=1}^{\infty} c_n \lim_{z\to z_0} \frac{z^n - z_0^n}{z - z_0} = \sum_{n=1}^{\infty} \lim_{z\to z_0} u_n(z) \overset{\text{uniform conv}}{=} \lim_{z\to z_0} \sum_{n=1}^{\infty} u_n(z) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$$

which completes our proof.

Corollary Let $\{c_n\}_{n=0}^{\infty}$ be a sequence of complex numbers and let $a \in \mathbb{C}$. Then

• the power series centered in a, $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$ has derivatives of all orders on the open convergence disc $D_R(a)$, with derivatives given by

$$f^{(p)}(z) = \sum_{n=p}^{\infty} \frac{n!}{(n-p)!} c_n (z-a)^{n-p} \quad \text{for each } p = 1, 2, \ldots, \tag{9}$$

• In particular, we observe that

$$c_p = \frac{f^{(p)}(a)}{p!} \quad \text{for each } p = 0, 1, 2, \ldots \tag{10}$$

In other words, a power series is the Taylor series of its sum.

The proof is a straightforward consequence of the previous theorem by setting $\zeta = z - a$, $g(\zeta) = f(\zeta + a)$ and a recursion to compute the expression for the successive derivatives.

Important Remarks

• This corollary shows that *if two power series centered at the same point a are equal in a neighborhood of a, they are equal term by term*.

• We will show in this course that *every analytic function on an open set can be locally expanded in a power series*. In other words, every analytic function on an open set has derivatives of all orders.

This is one of the most remarkable results of the theory of complex functions.

§ 3 The Exponential, Trigonometric, and Logarithmic Functions

3.1 The Exponential

• The exponential function is defined, on a domain to be defined later, as the function which satisfies

$$\begin{cases} \dfrac{df}{dz} = f(z) \\ f(0) = 1 \end{cases} \tag{11}$$

Let us assume it has a power series expansion $f(z) = \sum_{n=0}^{\infty} c_n z^n$ on some disc of convergence. Then

$$f'(z) = \sum_{n=1}^{\infty} n c_n z^{n-1} = f(z) = \sum_{n=0}^{\infty} c_n z^n \quad \text{and} \quad f(0) = 1$$

implies that

$$\begin{cases} c_{n-1} = n c_n & \text{if } n \geq 1 \\ c_0 = 1 & \text{by the initial condition } f(0) = 1 \end{cases} \implies c_n = \frac{c_{n-1}}{n} = \cdots = \frac{c_0}{n!} = \frac{1}{n!}$$

So if f exists and has a power series, this series is

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Also since

$$\left(\frac{1}{n!}\right)^{1/n} = \left(\prod_{k=1}^{n} \frac{1}{k}\right)^{1/n} \leq \frac{1}{n} \sum_{k=1}^{n} \frac{1}{k} \leq \frac{1}{n}\left(1 + \int_1^n \frac{1}{t}\, dt\right) = \frac{1 + \ln n}{n} \text{ for } n \geq 1 \implies \lim_{n\to\infty} (1/n!)^{1/n} = 0$$

the series above is indeed convergent on all of $\mathbb{C}$ which implies that the exponential function

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \tag{12}$$

is an entire function.

Elementary Properties of e^z :

• Let c be a complex number. Using the definition Eq.(11), $\frac{d}{dz}\left(e^z \cdot e^{c-z}\right) = e^z \cdot e^{c-z} + e^z \cdot (-e^{c-z}) = 0$ for all $z \in \mathbb{C}$, we have $e^z e^{c-z} = e^c$. In particular, we have $e^z e^{-z} = 1$ by setting $c = 0$, which implies also that $e^z \neq 0$. And we get $e^{a+b} = e^a \cdot e^b$ by setting $z = a$, $c = a + b$ for any complex numbers a and b. Note that we can also get $e^{a+b} = e^a \cdot e^b$ by using the power series expression Eq.(12)

$$\begin{aligned} e^{a+b} &= \sum_{n=0}^{\infty} \frac{(a+b)^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{a^k b^{n-k}}{k!(n-k)!} \\ &= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{a^k b^{n-k}}{k!(n-k)!} = \sum_{k=0}^{\infty} \frac{a^k}{k!} \sum_{n=k}^{\infty} \frac{b^{n-k}}{(n-k)!} = e^a e^b \end{aligned}$$

• Since the coefficients c_n of the power series for e^z are all real, $\overline{e^z} = e^{\bar{z}}$.

Thus, for each $(x, y) \in \mathbb{R}^2$,

$$\left|e^{x+iy}\right| = \sqrt{e^{x+iy} \cdot \overline{e^{x+iy}}} = \sqrt{e^{x+iy} \cdot e^{x-iy}} = \sqrt{e^{2x}} = e^x.$$

The Hyperbolic Functions

cosh and sinh are defined by

$$\cosh z = \frac{e^z + e^{-z}}{2} \quad , \quad \sinh z = \frac{e^z - e^{-z}}{2} \tag{13}$$

from which it is immediately clear that they are entire functions, with power series

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \quad , \quad \sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$$

From the definition (13), it is straightforward to prove that $\forall\ (z,\ a,\ b) \in \mathbb{C}^3$

$$\begin{aligned}
&\cosh^2 z - \sinh^2 z = 1\\
&\frac{d}{dz}(\sinh z) = \cosh z\\
&\frac{d}{dz}(\cosh z) = \sinh z\\
&\cosh(a+b) = \cosh a \cosh b + \sinh a \sinh b\\
&\sinh(a+b) = \sinh a \cosh b + \cosh a \sinh b
\end{aligned}$$

3.2 The Trigonometric Functions

cos and sin are defined on $\mathbb{C}$ by

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad , \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i} \tag{14}$$

which means that cos and sin are entire, with power series

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \quad , \quad \sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

From the definition (14), one immediately gets the well-known **Euler formula**:

$$e^{iz} = \cos z + i \sin z \quad \text{for all } z \in \mathbb{C} \tag{15}$$

Using (14), one can also rapidly rederive all the well-known formulae for the derivatives of cos and sin, $\cos(a+b)$, $\sin(a+b)\ldots$.

3.4 The Logarithm

Definition The logarithm function ln is defined such that $z = \ln w \iff e^z = w$, i.e. $z = \ln w$ is a root of the equation $e^z = w$. Since $e^z \neq 0\ \forall\, z \in \mathbb{C}$, the number 0 does not have a logarithm.

- Now, for $w \neq 0$, we may write $z = x + iy$ and

$$e^{x+iy} = w \iff \begin{cases} e^x &= |w| \\ e^{iy} &= \dfrac{w}{|w|} \end{cases}$$

  - The first equation in the system has a unique solution, since exp is a bijection from $\mathbb{R}$ to $\mathbb{R}_+$:

$$x = \ln |w|$$

- Let $s = w/|w|$. Since $|s| = 1$, the second equation $s = e^{iy}$ has a unique solution $y_0 \in [0, 2\pi)$, and infinitely many solutions $y = y_0 + 2\pi k$, $k \in \mathbb{Z}$.

We conclude that *every nonzero complex number has infinitely many logarithms, which differ from each other by integer multiples of* $2\pi i$.

If we write $w = re^{i\theta}$, $x = \ln r = \ln |w|$, and $y = \theta = \arg w$. Thus, for $w \neq 0$, we may write

$$\ln w = \ln |w| + i\arg w$$

and at the risk of repeating ourselves, the logarithm function is not single valued, because arg *is not single valued*.

- One usually adopts the convention that if $w \in \mathbb{R}_+$, $\ln w$ *is the bijective real logarithm*.

If $(a, b) \in \mathbb{C}^* \times \mathbb{C}$, we define

$$a^b := e^{b \ln a}$$

We see that a^b *is unique if* $a \in \mathbb{R}_+$*, according to our convention* regarding the logarithm of a strictly positive real number, but in general has multiple values, which differ by $\exp(2\pi k b)$, $k \in \mathbb{Z}$.

a^b *will be unique independently of* a *when* $b \in \mathbb{Z}$, as it should for integer powers of a complex number.

- One often likes to make the complex logarithm function single-valued by defining it as follows:

$$\text{Ln} \; : z \in \mathbb{C} \mapsto \text{Ln}\, z = \ln |z| + i\text{Arg}\, z \tag{16}$$

where $\text{Arg}\, z$ is the argument of z in the interval $(-\pi, \pi]$. Ln defined above is single valued, but the price to pay for this is that Ln is not continuous across the negative real axis. Ln is however analytic on $\mathbb{C} \setminus (\mathbb{R}_- \cup \{0\})$.

Ln is called *the principal branch of the logarithm*. In general, *a branch of a multiple-valued function* f *is any single-valued function* F *that is analytic in some domain at each point* z *of which the value* $F(z)$ *is one of the values of* f. The negative real axis is called a *branch cut* of this function.

One can construct infinitely many branches of ln by restricting the argument θ to be in the range $\alpha < \theta \leq \alpha + 2\pi$ with $\alpha \in [-\pi, \pi)$. The branch cuts are then the rays $\theta = \alpha$ including the origin. The origin is a point that is common to all the branch cuts, and is therefore called a *branch point* of the logarithm.

- Observe that $\ln e^z = z + i2\pi k$, $k \in \mathbb{Z}$

Even if ln is restricted to its principal branch Ln,

$$\text{Ln}\, e^z \neq z \text{ unless } Im\,(z) \in (-\pi, \pi]$$

Likewise, all we can say for an addition theorem is

$$\text{there exists an } k \in \mathbb{Z} \text{ such that } \ln\,(z_1 z_2) = \ln z_1 + \ln z_2 + i2\pi k$$

Example

$$\text{Ln}\,(-2i) = \ln 2 - i\frac{\pi}{2}$$

$$\text{Ln}\,(-2) + \text{Ln}\,(i) = \ln 2 + i\pi + i\frac{\pi}{2} = \ln 2 + i\frac{3\pi}{2} \neq \text{Ln}\,(-2i)$$

We close this lecture by saying that the inverse trigonometric functions arccos and arcsin can also be defined in terms of the logarithm, through the exponential definitions of cos and sin seen previously.